

SKEW POLYNOMIAL ALGEBRAS WITH COEFFICIENTS IN KOSZUL ARTIN-SCHELTER REGULAR ALGEBRAS

JI-WEI HE, FRED VAN OYSTAEYEN AND YINHUO ZHANG

ABSTRACT. Let A be a Koszul Artin-Schelter regular algebra with Nakayama automorphism ξ . We show that the Yoneda Ext-algebra of the skew polynomial algebra $A[z; \xi]$ is a trivial extension of a Frobenius algebra. Then we prove that $A[z; \xi]$ is Calabi-Yau; and hence each Koszul Artin Schelter regular algebra is a subalgebra of a Koszul Calabi-Yau algebra. A superpotential \hat{w} is also constructed so that the Calabi-Yau algebra $A[z; \xi]$ is isomorphic to the derivation quotient of \hat{w} . The Calabi-Yau property of a skew polynomial algebra with coefficients in a PBW-deformation of a Koszul Artin-Schelter regular algebra is also discussed.

INTRODUCTION

Let \mathbb{k} be a field of characteristic zero. Let $A = \bigoplus_{k \in \mathbb{Z}} A_k$ be a (\mathbb{Z}) -graded (\mathbb{k}) -algebra, and $M = \bigoplus_{k \in \mathbb{Z}} M_k$ be a graded left A -module. The n^{th} shift of M is the graded A -module $M(n)$ whose k^{th} component is: $M(n)_k = M_{n+k}$. If M is a graded A -bimodule and σ, φ are graded automorphisms of A , then ${}_{\sigma}M_{\varphi}$ is the graded A -bimodule whose left A -action is twisted by σ and right A -action is twisted by φ . A graded algebra A is called *Calabi-Yau* of dimension d , if (cf. [Gin]):

- (i) A is homologically smooth; that is, A has a bounded resolution of finitely generated graded projective A -bimodules;
- (ii) $\text{Ext}_{A^e}^i(A, A \otimes A) = 0$ if $i \neq d$ and $\text{Ext}_{A^e}^d(A, A \otimes A) \cong A(l)$ for some integer l as A -bimodules, where $A^e = A \otimes A^{op}$ is the enveloping algebra of A .

Calabi-Yau algebras are strongly related to Artin-Schelter (AS, for short) regular algebras. Recall that a connected graded algebra $A = \mathbb{k} \oplus A_1 \oplus A_2 \oplus \cdots$ is called an *AS-regular* algebra if (i) A has finite global dimension d ; (ii) $\underline{\text{Ext}}_A^i({}_A \mathbb{k}, A) = 0$ if $i \neq d$, and $\dim \underline{\text{Ext}}_A^d({}_A \mathbb{k}, A) = 1$. Here $\underline{\text{Ext}}$ is the derived functor of graded $\underline{\text{Hom}}$ (cf. [Sm]). If a graded Calabi-Yau algebra is also connected, then it must be AS-regular [BT]. If an AS-regular algebra A of global dimension d is Noetherian or Koszul, then A differs from a Calabi-Yau algebra by an automorphism; more precisely, we have $\text{Ext}_{A^e}^i(A, A \otimes A) = 0$ for $i \neq d$, and $\text{Ext}_{A^e}^d(A, A \otimes A) \cong A_{\xi}(l)$ for some graded automorphism ξ of A [VdB, BM]. We call the automorphism ξ the *Nakayama automorphism* of A .

Berger and Pichereau recently constructed in [BP] an interesting class of Calabi-Yau algebras of dimension 3, which are related to deformations of Poisson algebras. Given an AS-regular algebra A of global dimension 2 (which must be Koszul), Dubois-Violette showed in [DV] (also see [Z]) that A is defined by an invertible matrix M , that is, $A \cong T(V)/(f)$ with V a finite dimensional vector space with a basis $\{x_1, \dots, x_n\}$ and $f = (x_1, \dots, x_n)M(x_1, \dots, x_n)^t$. Here the matrix multiplications

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should be regarded in $T(V)$. Given such an AS-regular algebra, Berger and Pichereau constructed a graded algebra $B(f)$ which is generated by $V \oplus \mathbb{k}z$, and whose generating relations are cyclic partial derivations of $w = fz$. They proved that $B(f)$ is a Calabi-Yau algebra of dimension 3, and gave the classification, up to isomorphisms, of the obtained Calabi-Yau algebras. They also showed that $B(f)$ is isomorphic to the skew polynomial algebra $A[z; \xi]$ for some automorphism ξ of A . We find that ξ is exactly the Nakayama automorphism of A , and the Calabi-Yau property of $A[z; \xi]$ holds for general Koszul AS-regular algebra A by inspecting the Yoneda-Ext algebra of $A[z; \xi]$. The main results of this paper are the following (cf. Theorem 3.3).

Theorem 0.1. *Let A be a Koszul AS-regular algebra of global dimension d , and ξ the Nakayama automorphism of A . Then the skew polynomial algebra $B = A[z; \xi]$ is a Calabi-Yau algebra of dimension $d + 1$.*

Since an AS-regular algebra of global dimension 2 is always Koszul, our main results provide a new proof of [BP, Theorem 2.10] and our proof is totally different from that in [BP].

It was shown in [DV] and [BSW] that a Koszul AS-regular algebra A is determined by a twisted superpotential w . We show that the twisted superpotential w can be symmetrized into a superpotential \widehat{w} by introducing a new indeterminate, so that the skew polynomial algebra $A[z; \xi]$ is isomorphic to the derivation quotient algebra obtained from the superpotential \widehat{w} (see Theorem 4.4).

Let A be a Koszul AS-regular algebra, and ξ the Nakayama automorphism of A . A PBW-deformation of A is a filtered algebra U such that its associated graded algebra is isomorphic to A . For a PBW-deformation U of A , U has a filtration-preserving automorphism ζ such that $gr(\zeta) = \xi$, still called a Nakayama automorphism (in this case, ζ is not unique, see more details in Section 5). It is natural to ask whether $U[z; \zeta]$ is a Calabi-Yau algebra. Recall that a nongraded algebra U is *Calabi-Yau* of dimension d if (i) U is homologically smooth; (ii) $\text{Ext}_{U^e}^i(U, U \otimes U) = 0$ if $i \neq d$ and $\text{Ext}_{U^e}^d(U, U \otimes U) \cong U$ as U -bimodules.

Since a Nakayama automorphism ζ respects the filtration of U , we see that $U[z; \zeta]$ is in fact a PBW-deformation of $A[z; \xi]$, which is a Koszul algebra. Then we can use the techniques developed in [PP, Po] for PBW-deformations of Koszul algebras to discuss the Calabi-Yau property of $U[z; \zeta]$.

Now let A be a Koszul Calabi-Yau algebra, and U be a PBW-deformation of A . We may choose a specific Nakayama automorphism ζ of U (see Proposition 5.5) so that we have (cf. Theorem 5.8):

Theorem 0.2. *$U[z; \zeta]$ is Calabi-Yau.*

In the theorem above, if A is only an AS-regular algebra, then the result may fail. Counterexamples may be found in the case where A is AS-regular of global dimension 2. At the end of the paper, we provide a necessary and sufficient condition for $U[z; \zeta]$ to be Calabi-Yau with U a PBW-deformation of an AS-regular algebra of global dimension 2 (cf. Theorem 5.10).

1. TRIVIAL EXTENSIONS

Let $E = \mathbb{k} \oplus E_1 \oplus E_2 \oplus \cdots$ be a connected graded algebra, and M a graded E -bimodule. Recall that the trivial extension of E by M is the graded algebra $\Gamma(E, M) = E \oplus M$ with the product

$(x_1, m_1) * (x_2, m_2) = (x_1 x_2, x_1 \cdot m_2 + m_1 \cdot x_2)$ for $x_i \in E$ and $m_i \in M$. If $M_i = 0$ for all $i \leq 0$, then $\Gamma(E, M)$ is a connected graded algebra with the i^{th} component $\Gamma(E, M)_i = E_i \oplus M_i$.

We focus on trivial extensions of finite dimensional algebras. Let E be a finite dimensional connected graded algebra. We say that E is of *length* d if $E_d \neq 0$ and $E_i = 0$ for all $i > d$. Let E be a connected finite dimensional algebra of length d , and σ a graded automorphism of E . Let E^* be the dual vector space of E . Then E^* is a graded E -bimodule with the induced E -action. Let E_σ^* be the graded E -bimodule obtained from E^* with the right E -action twisted by σ . Given an integer $n > d$, consider the trivial extension of E by the bimodule $E_\sigma^*(-n)$: $\Gamma(E, E_\sigma^*(-n)) = E \oplus E_\sigma^*(-n)$. For simplicity, we write $\Gamma(E, \sigma, n)$ for $\Gamma(E, E_\sigma^*(-n))$. Now the product of $\Gamma(E, \sigma, n)$ is defined by: $(x_1, f_1) * (x_2, f_2) = (x_1 x_2, x_1 \cdot f_2 + f_1 \cdot \sigma(x_2))$ for $x_1, x_2 \in E$ and $f_1, f_2 \in E^*$.

For later discussions, we introduce first some terminology. A connected graded algebra E of length d is called a graded Frobenius algebra if there is an isomorphism of graded left A -modules $\Theta : E \cong E^*(-d)$, or equivalently, there is a nondegenerated graded bilinear form $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathbb{k}(-d)$ such that $\langle x, yz \rangle = \langle xy, z \rangle$ for all $x, y, z \in E$. If E is graded Frobenius, then there is a unique graded automorphism φ of E such that $\Theta : E_\varphi \rightarrow E^*(-d)$ is an isomorphism of A -bimodules. The isomorphism φ is called the *Nakayama automorphism* of E . For the bilinear form, we have $\langle x, y \rangle = \langle y, \varphi(x) \rangle$. A graded Frobenius algebra E of length d is said to be *graded symmetric*, if $\langle x, y \rangle = (-1)^{i(d-i)} \langle y, x \rangle$ for all $x \in E_i$ and $y \in E_{d-i}$. In this case, $\varphi = \epsilon^{d-1}$, where $\epsilon : E \rightarrow E$ is defined by $\epsilon(x) = (-1)^i x$ for $x \in E_i$.

Proposition 1.1. *Let E be a connected graded algebra of length d , and σ a graded automorphism of E . Then the trivial extension $\Gamma(E, \sigma, n)$ ($n > d$) is a graded Frobenius algebra of length n , and the Nakayama automorphism φ of $\Gamma(E, \sigma, n)$ is given by $\varphi(x, f) = (\sigma^{-1}(x), f \circ \sigma)$ for all $x \in E$ and $f \in E^*$.*

Proof. Define a bilinear form $\langle \cdot, \cdot \rangle : \Gamma(E, \sigma, n) \times \Gamma(E, \sigma, n) \rightarrow \mathbb{k}(-n)$ by $\langle (x_1, f_1), (x_2, f_2) \rangle = f_2(x_1) + f_1(\sigma(x_2))$ for $x_1, x_2 \in E$ and $f_1, f_2 \in E^*$. A straightforward verification shows that $\langle (x_1, f_1), (x_2, f_2) * (x_3, f_3) \rangle = \langle (x_1, f_1) * (x_2, f_2), (x_3, f_3) \rangle$. Obviously, the bilinear form $\langle \cdot, \cdot \rangle$ is nondegenerated. Hence $\Gamma(E)$ is a Frobenius algebra. Moreover, $\langle (x_1, f_1), (x_2, f_2) \rangle = f_2(x_1) + f_1(\sigma(x_2)) = f_2(\sigma \circ \sigma^{-1}(x_1)) + f_1 \circ \sigma(x_2) = \langle (x_2, f_2), (\sigma^{-1}(x_1), f_1 \circ \sigma) \rangle$. Hence the Nakayama automorphism φ of $\Gamma(E, \sigma, n)$ is defined by $\varphi(x, f) = (\sigma^{-1}(x), f \circ \sigma)$ for all $x \in E$ and $f \in E^*$. \square

Remark 1.2. In Proposition 1.1, if we choose $\sigma = \epsilon^{n-1}$, then the Nakayama automorphism φ is given as follows: for $x \in E_i$ and $f \in E_i^*$, $\varphi(x, f) = ((-1)^{(n-1)i} x, f \circ \epsilon^{(n-1)i}) = ((-1)^{(n-1)i} x, (-1)^{(n-1)i} f)$. Hence $\varphi = \epsilon^{n-1}$. Therefore, $\Gamma(E, \epsilon^{n-1}, n)$ is a graded symmetric algebra.

Recall that a (cochain) differential graded algebra (dga, for short) (E, δ_E) is a graded algebra $E = \bigoplus_{n \in \mathbb{Z}} E_n$ together with a derivation δ_E of degree 1 such that $\delta_E^2 = 0$. A left differential graded module ${}_E M$ is a left graded E -module together with a differential δ_M such that $\delta_M(xm) = \delta_E(x)m + (-1)^{|x|} x \delta_M(m)$ for all homogeneous elements $x \in E$ and $m \in M$, where $|x|$ denotes the degree of x . Similarly, one has right differential graded modules and differential graded bimodules.

A *curved differential graded algebra* (cdga, for short) (cf. [PP, Po]) is a triple (E, δ_E, θ_E) , where E is a graded algebra, δ_E is a derivation of degree 1 and θ_E is a special element in E_2 , such that $\delta_E(\theta_E) = 0$ and $\delta_E^2(x) = \theta_E x - x \theta_E$ for all homogeneous elements $x \in E$. The element θ_E is

usually called the *curvature element* of E . Let $(E', \delta_{E'}, \theta_{E'})$ be another cdga. A *cdga morphism* $f : E \rightarrow E'$ is a graded algebra morphism such that $f(\theta_E) = \theta_{E'}$ and $f\delta_E = \delta_{E'}f$ (warning: our definition of cdga morphism given here is more restricted than that in [PP]). A *cdg E -bimodule* is a graded E -bimodule M endowed with a differential δ_M such that $\delta_M^2(m) = \theta_E m - m\theta_E$. Note that if the curvature element is zero, then a cdga is just a usual dga, and a cdg bimodule is a usual dg bimodule.

Let (E, δ_E, θ_E) be a cdga, and let M be a cdg E -bimodule. The trivial extension of E by M is the cdga $(\Gamma_{cdg}(E, M), \delta_{\Gamma_{cdg}}, \theta_{\Gamma_{cdg}})$ defined as follows: as a graded algebra $\Gamma_{cdg}(E, M)$ is just the trivial extension $\Gamma(E^\sharp, M^\sharp)$, where E^\sharp is the underlying graded algebra by forgetting the derivation δ_E of E , and M^\sharp is the underlying graded module of M ; the derivation $\delta_{\Gamma_{cdg}}$ is defined by

$$\delta_{\Gamma_{cdg}}(x, m) = (\delta_E(x), \delta_M(m))$$

for all $x \in E$ and $m \in M$; and the curvature element $\theta_{\Gamma_{cdg}} = (\theta_E, 0)$.

Let $M(n)$ be the n^{th} shift of the graded E -bimodule M . Note that the differential $\delta_{M(n)}$ and the E -actions of $M(n)$ should be changed slightly so that $M(n)$ is also a cdg E -bimodule: For a homogeneous element $m \in M$, we denote by $m(n)$ the corresponding element in $M(n)$. Then $\delta_{M(n)}(m(n)) = (-1)^n \delta_M(m)(n)$. Let $x \in E$ be a homogeneous element. The left E -action on $M(n)$ is defined by $x \diamond (m(n)) = (-1)^{n|x|}(x \cdot m)(n)$ and the right E -action is defined by $(m(n)) \diamond x = (m \cdot x)(n)$, where $x \cdot m$ and $m \cdot x$ are E -actions on M .

Let $M^\vee = \bigoplus_{n \in \mathbb{Z}} M_n^*$ be the graded dual of M . Then M^\vee is a cdg E -bimodule with the differential δ_{M^\vee} and E -actions defined as follows: for homogeneous elements $f \in M^\vee$, $m \in M$ and $x \in E$, we have

$$(1) \quad \delta_{M^\vee}(f) = (-1)^{|f|+1} f \circ \delta_M,$$

$$(2) \quad (x \rightharpoonup f)(m) = (-1)^{|x|(|f|+|m|)} f(m \cdot x) \text{ and } (f \leftharpoonup x)(m) = f(x \cdot m).$$

Now let $E = k \oplus E_1 \oplus \cdots \oplus E_d$ ($E_d \neq 0$) be a finite dimensional cdga with differential δ_E . Then $E^* = E^\vee$ is a cdg E -bimodule. Hence the trivial extension $\Gamma_{cdg}(E, E^*(d+1))$ is a cdga.

Let M be a cdg E -module. Note that $(M(n))^\sharp$ is different from $M^\sharp(n)$ as graded E^\sharp -bimodules. We remark that the graded algebra $\Gamma_{cdg}(E, E^*(-d-1))^\sharp$ is different from the trivial extension $\Gamma(E^\sharp, (E^*)^\sharp(-d-1))$ of the graded algebra E^\sharp . However, we have the following result.

Lemma 1.3. *As a graded algebra, we have*

$$\Gamma_{cdg}(E, E^*(-d-1))^\sharp = \Gamma(E^\sharp, {}_{\epsilon^d}(E^*)^\sharp(-d-1)).$$

Proof. As before, we denote by $x \cdot f$ and $f \cdot x$ for $x \in E$ and $f \in E^*$ the E^\sharp -actions on the graded dual $(E^*)^\sharp$. Note that in the definition (2) of the E -actions on E^* , $f(x \cdot m)$ is not zero only if $|x| + |m| = -|f|$. Hence we have $(x \rightharpoonup f) = (-1)^{|x|} x \cdot f$.

It suffices to verify the multiplication of the cdga $\Gamma_{cdg}(E, E^*(-d-1))$ is equal to the multiplication of $\Gamma(E^\sharp, {}_{\epsilon^d}(E^*)^\sharp(-d-1))$. Indeed, for homogeneous elements $x, y \in E$ and $f, g \in E^*$, we have

$$\begin{aligned} (x, f(d+1)) * (y, g(-d-1)) &= (xy, x \diamond (g(-d-1)) + (f(-d-1)) \diamond y) \\ &= (xy, (-1)^{|x|(d+1)}(x \rightharpoonup g)(-d-1) + (f \leftharpoonup y)(-d-1)) \\ &= (xy, (-1)^{|x|d}(x \cdot g)(-d-1) + (f \cdot y)(-d-1)) \\ &= (xy, (\epsilon^d(x) \cdot g)(-d-1) + (f \cdot y)(-d-1)) \end{aligned}$$

Now it is easy to see that the last item in the identities above is exactly the multiplication of elements $(x, f(d+1))$ and $(y, g(-d-1))$ in the trivial extension $\Gamma(E^\sharp, {}_{\epsilon^d}(E^*)^\sharp(-d-1))$. \square

2. YONEDA ALGEBRAS

In this section, we will compute the Yoneda products of a skew polynomial algebra with coefficients in a Koszul algebra. We first recall the definition of a Koszul algebra. Let V be a finite dimensional vector space. A *quadratic algebra* A is a connected graded algebra of form $A = T(V)/(R)$, where $R \subseteq V \otimes V$ and (R) is the two-sided ideal of $T(V)$ generated by R . The *quadratic dual* $A^!$ of a quadratic algebra A is defined to be $A^! = T(V^*)/(R^\perp)$, where $R^\perp \subseteq V^* \otimes V^*$ is the orthogonal complement of R . One easily sees that $(A^!)^! = A$. Let $\phi : A \rightarrow A$ be an automorphism of the quadratic algebra A . The restriction of ϕ to $A_1 = V$ induces a bijective linear map $f : V^* \rightarrow V^*$. Since A is quadratic, we see that f defines an automorphism $\phi^!$ of the quadratic dual algebra $A^!$. We call $\phi^!$ the *automorphism of $A^!$ dual to ϕ* . Since $(A^!)^! = A$, we have $(\phi^!)^! = \phi$.

A quadratic algebra A is called a *Koszul algebra* [Pr] if the trivial graded module ${}_A\mathbb{k}$ admits a graded projective resolution:

$$\cdots \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow {}_A\mathbb{k} \longrightarrow 0,$$

such that the graded module P^{-n} is generated in degree n for all $n \geq 0$. Recall that if A is Koszul, then the Yoneda algebra $E(A) := \bigoplus_{i \geq 0} \text{Ext}_A^i({}_A\mathbb{k}, {}_A\mathbb{k}) \cong A^!$. Moreover, A is Koszul if and only if $A^!$ is Koszul [Sm]. We refer to [Pr] and [Sm] for further properties of Koszul algebras.

Let $A = T(V)/(R)$ be a Koszul algebra, and let $C_0 = \mathbb{k}$, $C_{-1} = V$, $C_{-2} = R$ and $C_{-n} = \bigcap_{i=0}^{n-2} V^{\otimes i} \otimes R \otimes V^{n-i-2}$ for $n \geq 3$. Then $C = \bigoplus_{n \geq 0} C_{-n}$ is a graded subcoalgebra of the tensor coalgebra $T(V)$. Moreover, as graded algebras, $E(A) \cong C^\vee = \bigoplus_{n \geq 0} C_{-n}^* \cong A^!$.

Consider the graded minimal projective resolution of the trivial module ${}_A\mathbb{k}$:

$$(3) \quad \cdots \longrightarrow A \otimes C_{-n} \xrightarrow{\partial^{-n}} \cdots \xrightarrow{\partial^{-2}} A \otimes C_{-1} \xrightarrow{\partial^{-1}} A \longrightarrow {}_A\mathbb{k} \longrightarrow 0,$$

where the differential is given on pure tensors by:

$$\partial^{-n}(a \otimes x_1 \otimes \cdots \otimes x_n) = ax_1 \otimes x_2 \otimes \cdots \otimes x_n,$$

for all $a \in A$ and $x_1, \dots, x_n \in V$.

Let σ be a graded automorphism of A . Since A is Koszul, σ induces an automorphism (also denoted by σ) of C in the obvious way. Let σ^\vee be the automorphism of graded algebra C^\vee induced by σ . Since $A^! \cong C^\vee$, we see that $\sigma^\vee = \sigma^!$. Let $B = A[z; \sigma]$ be the algebra of skew polynomials with coefficients in A . We assume that z is of degree 1. Then it is well known that B is also a Koszul algebra (cf. [ST], for example). The elements of B are of the sums of the elements of the form az^i

with $i \geq 0$ and $a \in A$, moreover $za = \sigma(a)z$. We want to construct a minimal projective resolution of the trivial module ${}_B\mathbb{k}$. The following construction is standard (cf. [GS] or [Ph]).

Clearly, B is free both as a left A -module or as a right A -module. Applying the exact functor $B \otimes_A -$ to the projective resolution (3) of \mathbb{k} , we obtain the following complex:

$$(4) \quad \cdots \longrightarrow B \otimes C_{-n} \longrightarrow \cdots \longrightarrow B \otimes C_{-1} \longrightarrow B \longrightarrow 0.$$

The complex is exact except at the final position. The cohomology at the final position is B/BA_+ . By abusing the notation, we also denote the differential of the complex (4) by ∂ .

For each $n \geq 0$, we define a homomorphism of left B -modules

$$f^{-n} : B \otimes C_{-n} \longrightarrow B \otimes C_{-n}$$

by

$$f^{-n}(1 \otimes x_1 \otimes \cdots \otimes x_n) = z \otimes \sigma^{-1}(x_1) \otimes \cdots \otimes \sigma^{-1}(x_n)$$

for all $x_1, \dots, x_n \in V$. It is easy to check that these f^{-n} are compatible with the differential of the complex (4). Hence $f = \prod_{n \geq 0} f^{-n}$ is a morphism of complexes. The mapping cone of f reads as follows:

$$(5) \quad \cdots \longrightarrow B \otimes C_{-n} \oplus B \otimes C_{-n+1} \xrightarrow{\delta^{-n}} \cdots \longrightarrow B \otimes C_{-2} \oplus B \otimes C_{-1} \xrightarrow{\delta^{-2}} B \otimes C_{-1} \oplus B \xrightarrow{\delta^{-1}} B \longrightarrow 0,$$

where the differential is given by: $\delta^{-n} = \begin{pmatrix} \partial^{-n} & f^{-n+1} \\ 0 & -\partial^{-n+1} \end{pmatrix}$ for $n \geq 2$, and $\delta^{-1} = (\partial^{-1}, f_0)$ for $n = 1$.

A straightforward verification shows that the complex (5) is exact except at the final position, and the cohomology at the final position is \mathbb{k} . Hence we have the following lemma.

Lemma 2.1. *The complex (5) is a minimal projective resolution of the trivial module ${}_B\mathbb{k}$.*

Next we compute the Yoneda product of $E(B)$. Note that $\text{Hom}_B(B \otimes C_{-n} \oplus B \otimes C_{-n+1}, {}_B\mathbb{k}) \cong C_{-n}^* \oplus C_{-n+1}^*$. For $\alpha \in C_{-n}^*$ and $\beta \in C_{-n+1}^*$, we view (α, β) as a homomorphism from $B \otimes C_{-n} \oplus B \otimes C_{-n+1}$ to ${}_B\mathbb{k}$. Consider the following diagram:

$$\begin{array}{ccccccc} \cdots B \otimes C_{-n-k} \oplus B \otimes C_{-n-k+1} & \xrightarrow{\delta^{-n-k}} & \cdots & \longrightarrow & B \otimes C_{-n} \oplus B \otimes C_{-n+1} & \longrightarrow & \cdots \\ \downarrow g_k & & & & \downarrow g_0 & \searrow (\alpha, \beta) & \\ \cdots B \otimes C_{-k} \oplus B \otimes C_{-k+1} & \xrightarrow{\delta^{-k}} & \cdots & \longrightarrow & B & \xrightarrow{\delta^{-1}} & \mathbb{k} \longrightarrow 0, \end{array}$$

where the graded B -module homomorphisms g_k 's are defined as follows: for $k \geq 1$,

$$g_k(1 \otimes x_1 \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_{n+k}, 0) = (1 \otimes x_1 \otimes \cdots \otimes x_k \alpha(x_{k+1} \otimes \cdots \otimes x_{n+k}), 0),$$

and:

$$\begin{aligned} & g_k(0, 1 \otimes x'_1 \otimes \cdots \otimes x'_k \otimes x'_{k+1} \otimes \cdots \otimes x'_{n+k-1}) \\ &= ((-1)^k 1 \otimes x'_1 \otimes \cdots \otimes x'_k \beta(x'_{k+1} \otimes \cdots \otimes x'_{n+k-1}), 1 \otimes x'_1 \otimes \cdots \otimes x'_{k-1} \alpha(\sigma^{-1}(x'_k) \otimes \cdots \otimes \sigma^{-1}(x'_{n+k-1}))); \end{aligned}$$

for $k = 0$,

$$g_0(1 \otimes x_1 \otimes \cdots \otimes x_n, 1 \otimes x'_1 \otimes \cdots \otimes x'_{n-1}) = \alpha(x_1 \otimes \cdots \otimes x_n)1 + \beta(x'_1 \otimes \cdots \otimes x'_{n-1})1.$$

A direct verification shows that the above diagram is commutative.

Since the projective resolution (5) is minimal, we have $\text{Ext}_B^n(A\mathbb{k}, A\mathbb{k}) \cong C_{-n}^* \oplus C_{-n+1}^*$ for all $n \geq 0$. Now assume $\alpha' \in C_{-k}^*$ and $\beta' \in C_{-k+1}^*$, we have:

$$(6) \quad (\alpha', \beta') * (\alpha, \beta) = (\alpha', \beta') \circ g_k = (\alpha' \cdot \alpha, (-1)^k \alpha' \cdot \beta + \beta' \cdot (\sigma^{-1})^!(\alpha)).$$

Proposition 2.2. *Let A be a Koszul algebra, σ a graded automorphism of A , and $B = A[z; \sigma]$. Then $E(B) \cong \Gamma(A^!, {}_{\epsilon}A_{\psi}^!(-1))$, where $\psi = (\sigma^{-1})^!$ is the automorphism of $A^!$ dual to σ^{-1} .*

Proof. Note that $A^! \cong C^\vee$ as graded algebras. The lemma is a direct consequence of the equation (6). \square

3. SKEW POLYNOMIAL ALGEBRAS WITH COEFFICIENTS IN KOSZUL ARTIN-SCHELTER REGULAR ALGEBRAS

In this section, A is a Koszul Artin-Schelter regular algebra of global dimension d . The Artin-Schelter regularity of A implies that $E(A) \cong A^!$ is a Frobenius algebra of length d [Sm]. Let φ be the Nakayama automorphism of the Frobenius algebra $A^!$.

The following result was originally proved by Van den Bergh in [VdB] in the Noetherian case. The result for general Koszul algebras was proved by Berger and Marconnet in [BM, Proof of Theorem 6.3].

Lemma 3.1. *Let $A = T(V)/(R)$ be a Koszul AS-regular algebra of global dimension d . Let φ be the Nakayama automorphism of $A^!$, and $\phi := \varphi^!$ the automorphism of A dual to φ . Then $\text{Ext}_{A^e}^i(A, A \otimes A) = 0$ for $i \neq d$, and*

$$\text{Ext}_{A^e}^d(A, A \otimes A) \cong A_\xi(d),$$

where $\xi = \epsilon^{d+1}\phi^{-1}$.

The automorphism ξ in the lemma is called the *Nakayama automorphism* of A .

The above lemma implies the following result (also see [HVZ1]).

Lemma 3.2. *Let A be a Koszul algebra. Then A is Calabi-Yau if and only if $E(A)$ is a graded symmetric algebra.*

Now we may prove the main result of this section.

Theorem 3.3. *Let A be a Koszul AS-regular algebra of global dimension d with the Nakayama automorphism ξ . Then the skew polynomial algebra $B = A[z; \xi]$ is a Calabi-Yau algebra of dimension $d + 1$.*

Proof. Keep the notions as in Lemma 3.1. By Proposition 2.2, we have $E(B) \cong \Gamma(A^!, {}_{\epsilon}A_{\psi}^!(-1))$, where $\psi = (\xi^{-1})^!$. Note that $\xi^{-1} = \epsilon^{d+1}\phi$. Then $\psi = (\xi^{-1})^! = \epsilon^{d+1}\varphi$. Therefore, we have ${}_{\epsilon}A_{\psi}^! \cong A_{\epsilon^d\varphi}^!$. Since $A^!$ is graded Frobenius with Nakayama automorphism φ , we have an isomorphism of $A^!$ -bimodules $A_{\varphi}^! \cong (A^!)^*(-d)$, which implies $A_{\epsilon^d\varphi}^! \cong (A^!)_{\epsilon^d}^*(-d)$ since $\epsilon\varphi = \varphi\epsilon$. Now we have:

$$(7) \quad E(B) \cong \Gamma(A^!, {}_{\epsilon}A_{\psi}^!(-1)) \cong \Gamma(A^!, (A^!)_{\epsilon^d}^*(-d-1)) = \Gamma(A^!, \epsilon^d, d+1).$$

By Remark 1.2, $E(B)$ is a graded symmetric algebra. Lemma 3.2 implies that B is a Calabi-Yau algebra since B is a Koszul algebra. \square

Corollary 3.4. *If A is a Koszul Calabi-Yau algebra, so is $A[z]$.*

Let V be a vector space of dimension n . Fix a basis $\{x_1, \dots, x_n\}$ of V , and x_1^*, \dots, x_n^* the dual basis of V^* . Given an invertible $n \times n$ matrix M , let $f = (x_1, \dots, x_n)M(x_1, \dots, x_n)^t$, where the matrix multiplications are in the tensor algebra $T(V)$. The quadratic algebra A has the following properties [DV, Z]:

- (i) A is a Koszul AS-regular algebra of global dimension 2;
- (ii) A is a domain;
- (iii) the quadratic dual $A^!$ is defined by the matrix M in the following way: there is a basis ϖ of $A_2^!$, such that for $\alpha = a_1x_1^* + \dots + a_nx_n^* \in A_1^!$ and $\beta = b_1x_1^* + \dots + b_nx_n^* \in A_1^!$, $\alpha\beta = \mathbf{a}M\mathbf{b}^t\varpi$, where $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{k}^n$ and $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{k}^n$ (cf. [HVZ2]);
- (iv) the Nakayama automorphism of $A^!$ is defined in the way: $\varphi(\varpi) = \varpi$ and

$$\varphi(x_1^*, \dots, x_n^*) = (x_1^*, \dots, x_n^*)M^{-1}M^t;$$

- (v) the Nakayama automorphism ξ of A is defined in the following way:

$$\xi(x_1, \dots, x_n) = -(x_1, \dots, x_n)M^tM^{-1}.$$

Let us check the generating relations of the skew polynomial algebra $B = A[z; \xi]$. Since B is generated by x_1, \dots, x_n, z and is quadratic, B is defined by the relations $f = 0$, $zx_1 = \xi(x_1)z$, \dots , $zx_n = \xi(x_n)z$. Now $zx_i = -(x_1, \dots, x_n)M^tM^{-1}(0, \dots, 1, \dots, 0)^t z$. If we put zx_1, \dots, zx_n into a column, we obtain

$$(8) \quad M^t \begin{pmatrix} zx_1 \\ \vdots \\ zx_n \end{pmatrix} = -M \begin{pmatrix} x_1z \\ \vdots \\ x_nz \end{pmatrix}.$$

Then we see that the skew algebra B above is isomorphic to the algebra $B(f)$ constructed in [BP], which is the algebra generated by x_1, \dots, x_n, z with relations $f = 0$ and equations in (8). As a corollary, we recover [BP, Theorem 2.10].

Theorem 3.5. [BP] *The graded algebra $B(f)$ is a Calabi-Yau algebra of dimension 3.*

Example 3.6. Recall that a Noetherian AS-regular algebra B of global dimension d is called a *quantum polynomial algebra* if B is a domain and has Hilbert series $H_B(t) = \frac{1}{(1-t)^d}$ (hence is Koszul). Let B be a Calabi-Yau quantum polynomial algebra of global dimension d . If B is \mathbb{Z}^2 -graded such that it is generated in degrees $(1, 0)$ and $(0, 1)$, and moreover $\dim B_{0,1} = 1$, then $B = A[z; \xi]$, where $A = \oplus_{n \geq 0} B_{n,0}$ is a quantum polynomial algebra of dimension $d - 1$ [KKZ, Proposition 3.5].

Let A be a Koszul AS-regular algebra of global dimension 2. By [DV, Z], there is a finite dimension vector space V with a fixed basis $\{x_1, \dots, x_n\}$ and an invertible $n \times n$ matrix M such that $A \cong T(V)/(f)$ where $f = (x_1, \dots, x_n)M(x_1, \dots, x_n)^t$. We already know that the Nakayama automorphism of A is defined by $\xi(x_1, \dots, x_n) = -(x_1, \dots, x_n)M^tM^{-1}$, and the Berger-Pichereau's algebra $B(f) \cong A[z; \xi]$. Let M' be another invertible $n \times n$ matrix, and $f' = (x_1, \dots, x_n)M'(x_1, \dots, x_n)^t$. Let $A' = T(V)/(f')$. Denote by ξ' the Nakayama automorphism of A' . The following result was proved in [BP] (indeed, Berger-Pichereau did not assume that M and M' are invertible).

Theorem 3.7. [BP] $B(f) \cong B(f')$ as graded algebras if and only if M and M' are cogredient; that is, there is an invertible $n \times n$ matrix P such that $M = PM'P^t$.

However, we do not know whether there is a similar result for general Koszul AS-regular algebras.

4. SUPERPOTENTIALS

Let V be a finite dimensional vector space. For the discussions in this section, we need additional notation. Let $\tau : V \otimes V \rightarrow V \otimes V$ be the usual twisting map. For $d \geq 2$, we set a sequence of maps: $\tau_d^0 = 1^{\otimes d} : V^{\otimes d} \rightarrow V^{\otimes d}$, $\tau_d^1 = \tau \otimes 1^{\otimes d-2}$, ..., $\tau_d^k = (1^{\otimes k-1} \otimes \tau \otimes 1^{\otimes d-k-1})\tau_d^{k-1}$ for all $k \geq 2$.

Let $\sigma : V \rightarrow V$ be a linear bijective map. Recall that an element $w \in V^{\otimes d}$ is called a *twisted superpotential of degree d* with respect to σ if

$$(9) \quad w = (-1)^{d-1} \tau_d^{d-1} \circ (\sigma \otimes 1^{\otimes d-1})(w).$$

If σ is the identity map, then w is called a *superpotential*.

Let $\psi : V \rightarrow \mathbb{k}$ be a linear map, and let $u \in V^{\otimes d}$. Following [BSW], we write:

$$[\psi u] = (\psi \otimes 1^{\otimes d-1})(u), \text{ and } [u\psi] = (1^{\otimes d-1} \otimes \psi)(u).$$

More generally, if $\Psi \in (V^*)^{\otimes k}$ ($k \leq d$), we have:

$$[\Psi u] = (\Psi \otimes 1^{\otimes d-k})(u), \text{ and } [u\Psi] = (1^{\otimes d-k} \otimes \Psi)(u).$$

One may check that an element $w \in V^{\otimes d}$ is a twisted superpotential with respect to σ if and only if, for all $\psi \in V^*$, $[\psi w] = (-1)^{d-1} [w(\psi \circ \sigma^{-1})]$.

For $\Psi \in V^{*\otimes k}$, define the partial derivation of a twisted superpotential w to be

$$\partial_\Psi(w) = [w\Psi].$$

Then $\partial_\Psi(w) \in V^{\otimes d-k}$. The *derivation quotient algebra* $A(w, k)$ of w is defined as follows [BSW]:

$$A(w, k) = T(V) / (\partial_\Psi(w) : \Psi \in V^{*\otimes k}).$$

Since in this paper we only discuss the quadratic derivation quotient algebra, we simply write $A(w)$ for $A(w, d-2)$ for a twisted superpotential w of degree d .

We now show that any twisted superpotential can be symmetrized into a superpotential by introducing an additional indeterminate. From the equation (9), we have the following facts.

Lemma 4.1. (i) If $i \geq j \geq 1$, then $\tau_d^i \circ \tau_d^j = \tau_d^{j-1} \circ (1 \otimes \tau_{d-1}^{i-1})$, and $\underbrace{\tau_d^{d-1} \circ \dots \circ \tau_d^{d-1}}_{d \text{ factors}} = 1$;

(ii) Let $w \in V^{\otimes d}$ be a twisted superpotential with respect to a bijection σ of V . Then we have

$$w = \sigma^{\otimes d}(w).$$

Proof. (i) is trivial. For the statement (ii), we have

$$\begin{aligned}
w &= (-1)^{d-1} \tau_d^{d-1} \circ (\sigma \otimes 1^{\otimes d-1})(w) \\
&= (-1)^{d-1} \tau_d^{d-1} \circ (\sigma \otimes 1^{\otimes d-1})((-1)^{d-1} \tau_d^{d-1} \circ (\sigma \otimes 1^{\otimes d-1})(w)) \\
&= (-1)^{2(d-1)} \tau_d^{d-1} \circ \tau_d^{d-1} \circ (\sigma^{\otimes 2} \otimes 1^{d-1})(w) \\
&\vdots \\
&= (-1)^{d(d-1)} \underbrace{\tau_d^{d-1} \circ \dots \circ \tau_d^{d-1}}_{d \text{ factors}} \circ \sigma^{\otimes d}(w) \\
&= \sigma^{\otimes d}(w). \quad \square
\end{aligned}$$

Proposition 4.2. Assume that $w \in V^{\otimes d}$ is a twisted superpotential with respect to a bijection σ of V . We construct an element $\hat{w} := \hat{w}(w, \sigma) \in (V \oplus \mathbb{k}z)^{\otimes d+1}$ as follows:

$$\hat{w} := \hat{w}(w, \sigma) = \sum_{i=0}^d (-1)^i \tau_{d+1}^i (1 \otimes \sigma^{\otimes i} \otimes 1^{\otimes d-i})(z \otimes w).$$

Then \hat{w} is a superpotential of degree $d+1$.

Proof. We need to show the identity: $\hat{w} = (-1)^d \tau_{d+1}^d(\hat{w})$. This follows the following computations:

$$\begin{aligned}
&\tau_{d+1}^d(\hat{w}) \\
&= \tau_{d+1}^d \left(\sum_{i=0}^d (-1)^i \tau_{d+1}^i (1 \otimes \sigma^{\otimes i} \otimes 1^{\otimes d-i})(z \otimes w) \right) \\
&= \tau_{d+1}^d(z \otimes w) + \sum_{i=1}^d (-1)^i \tau_{d+1}^{i-1} (1 \otimes \tau_d^{d-1})(1 \otimes \sigma^{\otimes i} \otimes 1^{\otimes d-i})(z \otimes w) \\
&= \tau_{d+1}^d(z \otimes w) + \sum_{i=1}^d (-1)^i \tau_{d+1}^{i-1} (1 \otimes \tau_d^{d-1})(1 \otimes 1 \otimes \sigma^{\otimes i-1} \otimes 1^{\otimes d-i})(1 \otimes \sigma \otimes 1^{\otimes d-1})(z \otimes w) \\
&= \tau_{d+1}^d(z \otimes w) + \sum_{i=1}^d (-1)^i \tau_{d+1}^{i-1} (1 \otimes \sigma^{\otimes i-1} \otimes 1^{\otimes d-i+1})(z \otimes \tau_d^{d-1}(\sigma \otimes 1^{\otimes d-1})(w)) \\
&= \tau_{d+1}^d(z \otimes w) + \sum_{i=1}^d (-1)^i \tau_{d+1}^{i-1} (1 \otimes \sigma^{\otimes i-1} \otimes 1^{\otimes d-i+1})(z \otimes (-1)^{d-1} w) \\
&= \tau_{d+1}^d(z \otimes w) + \sum_{i=1}^d (-1)^{d+i-1} \tau_{d+1}^{i-1} (1 \otimes \sigma^{\otimes i-1} \otimes 1^{\otimes d-i+1})(z \otimes w) \\
&= \tau_{d+1}^d(z \otimes w) + \sum_{j=0}^{d-1} (-1)^{d+j} \tau_{d+1}^j (1 \otimes \sigma^{\otimes j} \otimes 1^{\otimes d-j})(z \otimes w) \\
&= \tau_{d+1}^d(z \otimes \sigma^{\otimes d}(w)) + \sum_{j=0}^{d-1} (-1)^{d+j} \tau_{d+1}^j (1 \otimes \sigma^{\otimes j} \otimes 1^{\otimes d-j})(z \otimes w) \\
&= (-1)^d \hat{w}. \quad \square
\end{aligned}$$

Let $A = T(V)/(R)$ be a Koszul AS-regular algebra of global dimension d . It is established in [BSW] and [DV] that A is defined by a twisted superpotential; that is, $A \cong A(w)$ for some twisted superpotential w of degree d with respect to a suitable bijection $\sigma : V \rightarrow V$. We give a “visualized” description of the bijection σ and the twisted superpotential w .

Assume $\dim V = n$ and fix a basis $\{x_1, \dots, x_n\}$ of V . We use the definitions and notations as in Section 2. Since A is of global dimension d , we have $\dim C_{-d} = 1$ and $\dim C_{-d+1} = n$.

Choose a nonzero element $w \in C_{-d}$. Since $C_d = \bigcap_{i=0}^{d-2} V^{\otimes i} \otimes R \otimes V^{\otimes d-i-2}$, it follows that $w \in V \otimes C_{-d+1} \cap C_{-d+1} \otimes V$. We fix a basis of C_{-d+1} , say, $\{\theta_1, \dots, \theta_n\}$. Since $w \in V \otimes C_{-d+1}$, we may write w as $w = (x_1, \dots, x_n)M(\theta_1, \dots, \theta_n)^t$ for some $n \times n$ matrix M with entries in k . On the other hand, $w \in C_{-d+1} \otimes V$ implies that $w = (\theta_1, \dots, \theta_n)N(x_1, \dots, x_n)^t$ for some $n \times n$ matrix N . Let $\{x_1^*, \dots, x_n^*\}$ be the dual basis of V^* , and $\theta_1^*, \dots, \theta_n^*$ be the dual basis of C_{-d+1}^* . Since A is AS-regular, $A^! \cong C^\vee$ is a graded Frobenius algebra of length d . For $\alpha = (x_1^*, \dots, x_n^*)(a_1, \dots, a_n)^t \in V^* = A_1^!$ and $\beta = (\theta_1^*, \dots, \theta_n^*)(b_1, \dots, b_n)^t \in C_{-d+1}^* = A_{d-1}^!$, it is easy to see that the Yoneda product is given by:

$$\alpha * \beta = (a_1, \dots, a_n)M \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} w^*,$$

and

$$\beta * \alpha = (b_1, \dots, b_n)N \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} w^*.$$

Now the Frobenius property of $A^!$ implies that both M and N are invertible matrices. Let φ be the Nakayama automorphism of $A^!$. Then from the Yoneda product above, we see that

$$\varphi(x_1^*, \dots, x_n^*) = (x_1^*, \dots, x_n^*)N^{-1}M^t.$$

Let $\phi := \varphi^!$ be the automorphism of A dual to φ . Then the Nakayama automorphism of A is $\xi = \epsilon^{d+1}\phi^{-1}$, which acts on $A_1 = V$ as follows:

$$\xi(x_1, \dots, x_n) = (-1)^{d+1}(x_1, \dots, x_n)N^t M^{-1}.$$

We rewrite the element w in terms of the Nakayama automorphism ξ as follows:

$$\begin{aligned} w &= (\theta_1, \dots, \theta_n)N \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (\theta_1, \dots, \theta_n)M^t(M^{-1})^t N \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ &= (-1)^{d-1}(\theta_1, \dots, \theta_n)M^t \xi \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \end{aligned}$$

That is, $w = (-1)^{d-1}\tau_d^{d-1}(\xi \otimes 1^{\otimes d-1})(w)$.

Summarizing the above arguments, we obtain the following lemma.

Lemma 4.3. *Let $A = T(V)/(R)$ be a Koszul AS-regular algebra of dimension d with ξ the Nakayama automorphism. Then w is a twisted superpotential with respect to $\xi|_V$.*

Theorem 4.4. *Let $A = T(V)/(R)$ be a Koszul AS-regular algebra of global dimension $d \geq 2$ with ξ the Nakayama automorphism. Then*

- (i) $A \cong A(w)$, where w is a nonzero element in $\bigcap_{i=1}^{d-2} V^{\otimes i} \otimes R \otimes V^{d-2-i}$;
- (ii) $A[z; \xi] \cong A(\hat{w})$, where the superpotential $\hat{w} = \hat{w}(w, \xi)$ is formed in Proposition 4.2.

Proof. The statement (i) is essentially proved in [DV] and [BSW]. We include here our own proof. Assume $\dim R = m$. Since $A^!$ is Frobenius and $A_2^! = R^*$, we have that $A_{d-2}^! \cong C_{-d+2}^*$ is of dimension m . Fix a basis $\{r_1, \dots, r_m\}$ of R , and a basis $\{\vartheta_1, \dots, \vartheta_m\}$ of C_{-d+2} . As before, we let $\{r_1^*, \dots, r_m^*\}$ and $\{\vartheta_1^*, \dots, \vartheta_m^*\}$ be the dual bases of R^* and C_{-d+2}^* respectively. Note that we also have $w \in R \otimes C_{-d+2}$. Hence there is an $m \times m$ matrix L such that

$$(10) \quad w = (r_1, \dots, r_m)L(\vartheta_1, \dots, \vartheta_m)^t.$$

For $\alpha = (r_1^*, \dots, r_m^*)(a_1, \dots, a_m)^t$ and $\beta = (\vartheta_1^*, \dots, \vartheta_m^*)(b_1, \dots, b_m)^t$, we have

$$\alpha * \beta = (a_1, \dots, a_m)L(b_1, \dots, b_m)^t w^*.$$

By the Frobenius property of $A^!$, we have that L is invertible. Then from the expression of w as in (10), we see that $R = \{\partial_\Psi(w)|\Psi \in (V^*)^{\otimes d-2}\}$. Therefore $A \cong A(w)$.

(ii) Since w is a twisted superpotential with respect to ξ , \widehat{w} is a superpotential of degree $d+1$ by Proposition 4.2. Let $U = V \oplus \mathbb{k}z$. Then $\{x_0^* = z^*, x_1^*, \dots, x_n^*\}$ is a basis of U^* . Let us check the following facts:

- (a) $\{\partial_\Psi(\widehat{w})|\Psi \in (V^*)^{\otimes d-1}\} = \text{span}\{z \otimes x_i - \xi(x_i) \otimes z | i = 1, \dots, n\}$;
- (b) $R = \text{span}\{\partial_\Psi(\widehat{w})|\Psi = x_{i_1}^* \otimes \dots \otimes x_{i_{d-1}}^* \text{ at least one of } i_1, \dots, i_{d-1} \text{ is zero}\}$.

For $\Psi \in (V^*)^{\otimes d-1}$, we have

$$\partial_\Psi(\widehat{w}) = (1 \otimes 1 \otimes \Psi)[(z \otimes w) - \tau_{d+1}^1 \circ (1 \otimes \xi \otimes 1^{\otimes d-1})(z \otimes w)].$$

Recall that

$$w = (x_1, \dots, x_n)M \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}.$$

So, if we write $w = \sum_{i=1}^n x_i \otimes y_i$, then $(y_1, \dots, y_n) = (\theta_1, \dots, \theta_n)M^t$. Since M is invertible, we obtain that y_1, \dots, y_n are linear independent in $V^{\otimes d-1}$. Now we have

$$\partial_\Psi(\widehat{w}) = \sum_{i=1}^n (z \otimes x_i - \xi(x_i) \otimes z)\Psi(y_i).$$

Thus (a) follows.

For the identity (b), we choose $\Psi = x_{i_1}^* \otimes \dots \otimes x_{i_{d-2}}^* \otimes z^*$. Then $\partial_\Psi(\widehat{w}) = (-1)^d(1 \otimes 1 \otimes \Psi)(\xi^{\otimes d}(w) \otimes z)$. On the other hand, as we have seen that we may write w in the form (10). Again since L is invertible, we have

$$\text{span}\{\partial_\Psi(\widehat{w})|\Psi \text{ is of the form } x_{i_1}^* \otimes \dots \otimes x_{i_{d-2}}^* \otimes z^*\} = \text{span}\{(\xi \otimes \xi)(r_i)|i = 1, \dots, m\}.$$

As A is Koszul and ξ is the Nakayama automorphism of A , we have $\text{span}\{(\xi \otimes \xi)(r_i)|i = 1, \dots, m\} = R$. Since we obviously have $R \supseteq \text{span}\{\partial_\Psi(\widehat{w})|\Psi = x_{i_1}^* \otimes \dots \otimes x_{i_{d-1}}^* \text{ at least one of } i_1, \dots, i_{d-1} \text{ is zero}\}$, (b) follows.

Finally, since $R + \text{span}\{z \otimes x_i - \xi(x_i) \otimes z | i = 1, \dots, n\}$ is exactly the generating relations of $A[z; \xi]$, we have $A[z; \xi] \cong A(\widehat{w})$. \square

5. PBW-DEFORMATIONS

Let $A = A_0 \oplus A_1 \oplus \cdots$ be a positively graded algebra. Recall that a *PBW-deformation* of A is a filtered algebra U with an ascending filtration $0 \subseteq F_0U \subseteq F_1U \subseteq F_2U \subseteq \cdots$ such that the associated graded algebra $gr(U)$ is isomorphic to A . If $A = T(V)/(R)$ is a Koszul algebra, then a PBW-deformation U of A is determined by two linear maps $\nu : R \rightarrow V$ and $\theta : R \rightarrow \mathbb{k}$ in the sense that $U \cong T(V)/(r - \nu(r) - \theta(r) : r \in R)$ [BG, PP]. If $\theta = 0$, then we call U an *augmented* PBW-deformation of A . The dual map ν^* of the linear map $\nu : R \rightarrow V$ induces a derivation δ_{A^\dagger} on the dual algebra A^\dagger of A . If we view the linear map $\theta : R \rightarrow \mathbb{k}$ as an element in A_2^\dagger , then $(A^\dagger, \delta_{A^\dagger}, \theta)$ is a cdga. We call $(A^\dagger, \delta_{A^\dagger}, \theta)$ the *dual cdga* of U . Conversely, if there is a curved differential graded structure $(A^\dagger, \delta_{A^\dagger}, \theta)$, then the dual map of the linear map $\delta_{A^\dagger}|_{V^*} : V^* \rightarrow R^*$ and $\theta \in A_2^\dagger = R^*$ define a PBW-deformation of A [PP].

Now let A be a Koszul AS-regular algebra of global dimension d , and let U be a PBW-deformation of A . Assume that ξ is the Nakayama automorphism of A . The following lemma was proved by Yekutieli [Y] when A is Noetherian. For the general case, see [HVZ2].

Lemma 5.1. *We have $\text{Ext}_{U^e}^i(U, U \otimes U) = 0$ for $i \neq d$ and $\text{Ext}_{U^e}^d(U, U \otimes U) \cong U_\zeta$ as U -bimodules, where ζ is a filtration-preserving automorphism of U such that $gr(\zeta) = \xi$.*

The automorphism ζ in Lemma 5.1 is not unique. If ζ' is another automorphism of U such that the conditions in the lemma hold, then ζ' differs from ζ by an inner automorphism, that is, there is a unit $u \in U$ such that for all $a \in U$, $\zeta'(a) = u\zeta(a)u^{-1}$. Hence ζ is unique up to inner automorphisms. We call an automorphism ζ satisfying the conditions in Lemma 5.1 a *Nakayama automorphism* of U . Note that if A is a domain, then there is a unique automorphism satisfies the condition in Lemma 5.1. Hence in this case, we may say “the” Nakayama automorphism of U .

Next we discuss the Calabi-Yau property of the skew polynomial algebra $U[z; \zeta]$ with ζ a Nakayama automorphism of U .

The skew polynomial algebra $U[z; \zeta]$ is also a filtered algebra with filtration: $F_0U[z; \zeta] = \mathbb{k}$, $F_nU[z; \zeta] = \sum_{i+j=n} F_iUz^j$ for all $n > 0$ and $i, j \geq 0$. It is easy to see that $gr(U[z; \zeta]) \cong A[z; \xi]$. Hence we obtain:

Lemma 5.2. *$U[z; \zeta]$ is a PBW-deformation of $A[z; \xi]$.*

The following result was proved in [HVZ2].

Lemma 5.3. *Let B be a Koszul Calabi-Yau algebra of dimension d , and let $U = T(V)/(r - \nu(r) - \theta(r) : r \in R)$ be a PBW-deformation of B , and let $(B^\dagger, \delta_{B^\dagger}, \theta)$ be the cdga dual to U . If $\delta_{B^\dagger}(B_{d-1}^\dagger) = 0$, then U is a Calabi-Yau algebra.*

Conversely, if B is a domain and U is Calabi-Yau, then $\delta_{B^\dagger}(B_{d-1}^\dagger) = 0$.

Let $B = A[z; \xi]$. As we have already proved in Section 3, B is a Koszul Calabi-Yau algebra of dimension $d + 1$. To see whether $U[z; \zeta]$ is a Calabi-Yau algebra, it is sufficient to see whether the condition $\delta_{B^\dagger}(B_d^\dagger) = 0$ holds. By Proposition 2.2, $B^\dagger \cong \Gamma(A^\dagger, \epsilon A_{\psi}^\dagger(-1))$ where $\psi = (\xi^{-1})^\dagger$. We need to write out the differential on $\Gamma(A^\dagger, \epsilon A_{\psi}^\dagger(-1))$ induced by δ_{B^\dagger} through the previous isomorphism. The following lemma is trivial.

Lemma 5.4. *Let D and D' be quadratic algebras. If there are invertible linear maps $f : D_1 \rightarrow D'_1$ and $g : D_2 \rightarrow D'_2$ such that the following diagram commutes:*

$$\begin{array}{ccc} D_1 \otimes D_1 & \xrightarrow{f \otimes f} & D'_1 \otimes D'_1 \\ \downarrow \mu_D & & \downarrow \mu_{D'} \\ D_2 & \xrightarrow{g} & D'_2, \end{array}$$

where the vertical maps are multiplications of D and D' respectively, then the linear map f defines an isomorphism $\Phi : D \rightarrow D'$ in the following way: for any $x_1, \dots, x_n \in D_1$, $\Phi(x_1 x_2 \cdots x_n) = f(x_1) f(x_2) \cdots f(x_n)$.

Let us write down $B_1^!$ and $B_2^!$ of $B^!$ explicitly. Write $\widehat{V} = V \oplus \mathbb{k}z$. As before, we fix a basis $\{x_1, \dots, x_n\}$ for V , and let $\{x_1^*, \dots, x_n^*\}$ be the dual basis of V^* . Let $\tilde{r}_i = z \otimes \xi^{-1}(x_i) - x_i \otimes z$ for $i = 1, \dots, n$, and $\tilde{R} = \text{span}\{\tilde{r}_1, \dots, \tilde{r}_n\} \subseteq \widehat{V} \otimes \widehat{V}$. Then $B = T(\widehat{V})/(\tilde{R})$, where $\widehat{R} = R \oplus \tilde{R}$. Let $\{\tilde{r}_1^*, \dots, \tilde{r}_n^*\}$ be the dual basis of \tilde{R} and z^* be the element in \widehat{V}^* such that $z^*(z) = 1$ and $z^*(V) = 0$. We have $B_1^! = \widehat{V}^*$ and $B^! = \widehat{R}^* = R^* \oplus \tilde{R}^*$, equivalently $B_1^! = A_1^! \oplus \mathbb{k}z^*$ and $B_2^! = A_2^! \oplus \tilde{R}^*$.

Assume that the automorphism ξ of A acts on $A_1 = V$ as:

$$(11) \quad \xi(x_1, \dots, x_n) = (x_1, \dots, x_n)P$$

where $P = (p_{ij})$ is an invertible $n \times n$ matrix. Assume further $P^{-1} = (l_{ij})$. Then it is not hard to see that the product of two elements in $B_1^!$ is given as follows: for $x_i^*, x_j^* \in A_1^!$, the product $x_i^* \cdot x_j^*$ is just the product in $A^!$; $z^* \cdot z^* = 0$;

$$x_i^* \cdot z^* = -\tilde{r}_i^* \in \tilde{R}^* \subseteq B_2^! \text{ and } z^* \cdot x_i^* = \sum_{j=1}^n l_{ij} \tilde{r}_j^*.$$

Recall that the graded algebra $B^!$ is isomorphic to $\Gamma(A^!, {}_\epsilon A_\psi^!(-1))$ with $\psi = (\xi^{-1})^!$ (Proposition 2.2). We construct an isomorphism from $\Gamma(A^!, {}_\epsilon A_\psi^!(-1))$ to $B^!$ in detail. Note that $\Gamma(A^!, {}_\epsilon A_\psi^!(-1))_1 = A_1^! \oplus \mathbb{k}$ and $\Gamma(A^!, {}_\epsilon A_\psi^!(-1))_2 = A_2^! \oplus V^*$. We define linear maps $f : \Gamma(A^!, {}_\epsilon A_\psi^!(-1))_1 \rightarrow B_1^!$ and $g : \Gamma(A^!, {}_\epsilon A_\psi^!(-1))_2 \rightarrow B_2^!$ as follows: $f(x_i^*, 0) = x_i^*$ for all i and $f(0, 1) = z^*$; $g(\alpha, 0) = \alpha$ for all $\alpha \in A_2^!$ and $g(0, x_i^*) = \tilde{r}_i^*$ for all $i = 1, \dots, n$. Now one may easily check that the conditions of Lemma 5.4 above hold for f and g . Therefore f defines an isomorphism

$$(12) \quad \Phi : \Gamma(A^!, {}_\epsilon A_\psi^!(-1)) \rightarrow B^!$$

since both algebras are Koszul.

As before, let φ be the Nakayama automorphism of $A^!$, and let $\phi = \varphi^!$ be the automorphism of A dual to φ . Then $\xi = \epsilon^{d+1} \phi^{-1}$ by Proposition 3.1. Hence $\varphi(x_i^*) = (-1)^{d+1} (\xi^{-1})^!(x_i^*)$ for all $i = 1, \dots, n$. Since $A^!$ is Frobenius, there is an isomorphism of graded $A^!$ -bimodules $\Theta : A_\varphi^! \rightarrow A^{!*}(-d)$. Let $\varpi \in A_d^!$ be the element such that $\Theta(1)(\varpi) = 1$. Then ϖ is a basis of $A_d^!$. By the Frobenius property of $A^!$ again, we may choose elements $\omega_1, \dots, \omega_n$ in $A_{d-1}^!$ such that $x_i^* \omega_j = \delta_j^i \varpi$, where δ is the Kronecker delta function. Clearly, $\{\omega_1, \dots, \omega_n\}$ is a basis of $A_{d-1}^!$. Let ϖ^* and $\{\omega_1^*, \dots, \omega_n^*\}$ be the dual basis of the space $(A_d^!)^*$ and $(A_{d-1}^!)^*$ respectively. Consider the composition of the following isomorphisms:

$$h : {}_\epsilon A_\psi^! \xrightarrow{\epsilon^{d+1}} {}_\epsilon A_\varphi^! \xrightarrow{\Theta} {}_\epsilon A^{!*}(-d).$$

We have $h(1) = \varpi^*$ and $h(x_i^*) = \sum_{j=1}^n p_{ij} \omega_j^*$. The isomorphism h induces an isomorphism $\Gamma(A^!, {}_{\epsilon} A_{\psi}^!(-1)) \longrightarrow \Gamma(A^!, {}_{\epsilon^d} A^{!*}(-d-1))$. Combining this isomorphism with the inverse of Φ constructed in previous paragraph, we get an isomorphism of graded algebras:

$$(13) \quad \Psi : B^! \longrightarrow \Gamma(A^!, {}_{\epsilon^d} A^{!*}(-d-1)).$$

Now we have $\Psi(\alpha) = (\alpha, 0)$ for all $\alpha \in A^!$, and

$$(14) \quad \Psi(z^*) = (0, \varpi^*) \text{ and } \Psi(\tilde{r}_i^*) = (0, \sum_{j=1}^n p_{ij} \omega_j^*)$$

for all $i = 1, \dots, n$.

Since $U[z; \zeta]$ is a PBW-deformation of $B = A[z; \xi]$, to study the curved differential structure of $B^!$, we need to pick a specific Nakayama automorphism ζ . The following result was proved in [HVZ2].

Proposition 5.5. *Let $A = T(V)/(R)$ be a Koszul AS-Gorenstein algebra of global dimension d , and let $A^!$ be its dual algebra. Assume that $\{x_1, \dots, x_n\}$ is a basis of V , and $\{x_1^*, \dots, x_n^*\}$ is the dual basis of V^* .*

Let $U = T(V)/(r - \nu(r) - \theta(r) : r \in R)$ be a PBW-deformation of A , and let $(A^!, \delta_{A^!}, \theta)$ be the cdga dual to U . Choose a basis ϖ of $A_d^!$, and assume that $\omega_1, \dots, \omega_n$ is the basis of $A_{d-1}^!$ such that $x_i^ \omega_j = \delta_j^i \varpi$. Assume further $\delta_{A^!}(\omega_i) = \lambda_i \varpi$ for all $i = 1, \dots, n$. Then $\text{Ext}_{U^e}^i(U, U \otimes U) = 0$ for $i \neq d$, and*

$$\text{Ext}_{U^e}^d(U, U \otimes U) \cong U_{\zeta},$$

where the automorphism ζ acts on the generator as follows:

$$\zeta(x_i) = \xi(x_i) + \lambda_i.$$

Convention: From now on, ζ is the Nakayama automorphism of U as defined in Proposition 5.5.

Note that in Proposition 5.5, we view $V \oplus \mathbb{k}$ as a subspace of U through the obvious injective map. The scalars $\lambda_1, \dots, \lambda_n$ are independent of the choice of basis ϖ . In fact, if we choose another element ϖ' as a basis of $A_d^!$, then $\varpi' = k\varpi$ for some $k \in \mathbb{k}$. Hence $x_i^*(k\omega_j) = \delta_j^i \varpi'$ for all $i, j = 1, \dots, n$. Set $\omega'_i = k\omega_i$ for $i = 1, \dots, n$. Then $\omega'_1, \dots, \omega'_n$ is the basis of $A_{d-1}^!$ satisfying the condition in the proposition. Clearly, we have $\delta_{A^!}(\omega'_i) = k\lambda_i \varpi = \lambda_i \varpi'$.

Now we can write down the linear maps that determine the PBW-deformation $U[z; \zeta]$ of $A[z; \xi]$. Recall that $A[z; \zeta] \cong T(\widehat{V})/(\widehat{R})$ with $\widehat{V} = V \oplus \mathbb{k}z$ and $\widehat{R} = R \oplus \widetilde{R}$.

Lemma 5.6. $\widehat{U} := U[z; \zeta]$ viewed as a PBW-deformation of $B = A[z; \xi]$ is determined by the following linear maps:

$$\begin{aligned} \widehat{\nu} : R \oplus \widetilde{R} &\rightarrow V \oplus \mathbb{k}z, \widehat{\nu}(r) = \nu(r) \text{ for all } r \in R, \text{ and } \widehat{\nu}(\tilde{r}_i) = \lambda_i z \ (i = 1, \dots, n); \\ \widehat{\theta} : R \oplus \widetilde{R} &\rightarrow \mathbb{k}, \widehat{\theta}(r) = \theta(r) \text{ for all } r \in R, \text{ and } \widehat{\theta}(\tilde{r}_i) = 0 \ (i = 1, \dots, n). \end{aligned}$$

That is, $\widehat{U} \cong T(\widehat{V})/(\widehat{r} - \widehat{\nu}(\widehat{r}) - \widehat{\theta}(\widehat{r}) : \widehat{r} \in \widehat{R})$.

Proof. The lemma is clear since $z\xi^{-1}(x_i) - \lambda_i z = x_i z$ by Proposition 5.5, or equivalently, $z\xi^{-1}(x_i) - x_i z - \lambda_i z = 0$ in \widehat{U} . \square

Let us check the linear dual maps of $\widehat{\nu}$ and $\widehat{\theta}$. We have $\widehat{\nu}^* : \widehat{V}^* \rightarrow \widehat{R}^*$, $\widehat{\nu}^*|_{V^*} = \nu^*$, and $\widehat{\nu}^*(z^*) = \sum_{i=1}^n \lambda_i \widehat{r}_i^*$; $\widehat{\theta}^* = \theta^* : \mathbb{k} \rightarrow R^* \subseteq \widehat{R}^*$. Let $(A^!, \delta_{A^!}, \theta_{A^!})$ be the cdga dual to the PBW-deformation U of A . Since in the cdga $(B^!, \delta_{B^!}, \theta_{B^!})$, the differential $\delta_{B^!}$ is determined by $\widehat{\nu}^*$ and the curvature element $\theta_{B^!} = \widehat{\theta}$, we have

$$\delta_{B^!}(\alpha) = \delta_{A^!}(\alpha) \text{ for all } \alpha \in V^*$$

and

$$\delta_{B^!}(z^*) = \sum_{i=1}^n \lambda_i \widehat{r}_i^*.$$

The cdga structure on $B^!$ induces a cdga structure on the graded algebra $\Gamma(A^!, \epsilon^d(A^!)^*(-d-1))$ through the isomorphism Ψ as in (13) and (14). Denote by $\Gamma := \Gamma(A^!, \epsilon^d(A^!)^*(-d-1))$. Let $(\Gamma, \delta_\Gamma, \theta_\Gamma)$ be the cdga induced by $(B^!, \delta_{B^!}, \theta_{B^!})$. Then we have:

$$(15) \quad \theta_\Gamma = (\theta_{A^!}, 0);$$

$$(16) \quad \delta_\Gamma(\alpha, 0) = (\delta_{A^!}(\alpha), 0), \text{ for all } \alpha \in A^!$$

$$(17) \quad \delta_\Gamma(0, \varpi^*) = \left(0, \sum_{j=1}^n \lambda_j \left(\sum_{i=1}^n p_{ji} \omega_i^* \right) \right).$$

Proposition 5.7. *Let $A = T(V)/(R)$ be a Koszul AS-regular algebra of global dimension d with ξ the Nakayama automorphism, and $A^!$ be its quadratic dual algebra with φ the Nakayama automorphism. Let U be a PBW-deformation of A , and $(A^!, \delta_{A^!}, \theta_{A^!})$ be the cdga dual to U . If the composition $\epsilon^{d+1}\varphi$ is an automorphism of cdga $(A^!, \delta_{A^!}, \theta_{A^!})$, then*

(i) *The trivial extension $\Gamma_{\text{cdg}}(A^!, (A^!)^*(-d-1))$ of the cdga $(A^!, \delta_{A^!}, \theta_{A^!})$ is the dual cdga of the PBW-deformation $U[z; \zeta]$ of $A[z; \xi]$, where ζ is the Nakayama automorphism of U as in Proposition 5.5;*

(ii) *$U[z; \zeta]$ is a Calabi-Yau algebra.*

Proof. Keep the notions as before except we now choose the basis ϖ in Proposition 5.5 such that $\Theta(1)(\varpi) = 1$. Let us check the differential $\delta_{A^!}^*$ on $(A^!)^*$. Recall that $\delta_{A^!}(\omega_i) = \lambda_i \varpi$ for all $i = 1, \dots, n$ by assumption. Thus we have

$$\delta_{A^!}^*(\varpi^*) = \sum_{i=1}^n \lambda_i \omega_i^*.$$

Note that $\varphi = \epsilon^{d+1}(\xi^{-1})^!$, and ξ is represented as in (11). An easy computation shows that $\varphi(\varpi) = \varpi$ and that:

$$\varphi(\omega_i) = (-1)^{d+1} \sum_{j=1}^n p_{ji} \omega_j.$$

Now by assumption of the proposition $\delta_{A^!} \circ (\epsilon^{d+1}\varphi) = (\epsilon^{d+1}\varphi) \circ \delta_{A^!}$, we have $\delta_{A^!}^* \circ (\epsilon^{d+1}\varphi)^* = (\epsilon^{d+1}\varphi)^* \circ \delta_{A^!}^*$. Applying these morphisms to ϖ^* , we obtain:

$$(\epsilon^{d+1}\varphi)^* \circ \delta_{A^!}^*(\varpi^*) = (\epsilon^{d+1}\varphi)^* \left(\sum_{i=1}^n \lambda_i \omega_i^* \right) = \sum_{i=1}^n \lambda_i \sum_{j=1}^n p_{ij} \omega_j^*;$$

and

$$\delta_{A^!}^* \circ (\epsilon^{d+1}\varphi)^*(\varpi^*) = \delta_{A^!}^*(\varpi^*) = \sum_{i=1}^n \lambda_i \omega_i^*.$$

Hence we arrive at:

$$(18) \quad \sum_{i=1}^n \lambda_i \omega_i^* = \sum_{i=1}^n \lambda_i \sum_{j=1}^n p_{ij} \omega_j^*.$$

Comparing the equations (17) and (18), we see that the differential δ_Γ on $\Gamma := \Gamma(A^!, \epsilon^d(A^!)^*(-d-1))$, induced from the cdga $(B^!, \delta_{B^!}, \theta_{B^!})$, acts on the elements of degree 1 as:

$$(19) \quad \delta_\Gamma(\alpha, 0) = (\delta_{A^!}(\alpha), 0), \text{ for } \alpha \in A_1^!$$

$$(20) \quad \delta_\Gamma(0, \varpi^*) = \left(0, \sum_{i=1}^n \lambda_i \omega_i^*\right).$$

Since Γ is a quadratic algebra, the differential δ_Γ is determined by its action on the elements of degree 1. By Lemma 1.3, the underlying graded algebra of the trivial extension $\Gamma_{cdg}(A^!, (A^!)^*(-d-1))$ of the cdga $(A^!, \delta_{A^!}, \theta_{A^!})$ is exactly the graded algebra $\Gamma = \Gamma(A^!, \epsilon^d(A^!)^*(-d-1))$. Let $\delta_{\Gamma_{cdg}}$ be the differential of $\Gamma_{cdg}(A^!, (A^!)^*(-d-1))$. By a straightforward check we have $\delta_{\Gamma_{cdg}}(\alpha, 0) = (\delta_{A^!}(\alpha), 0)$, for $\alpha \in A_1^!$ and $\delta_{\Gamma_{cdg}}(0, z^*) = (0, \sum_{i=1}^n \lambda_i \omega_i^*)$. Comparing these equations with (19) and (20), we see that the cdga $\Gamma_{cdg}(A^!, (A^!)^*(-d-1))$ is isomorphic to the cdga $(\Gamma, \delta_\Gamma, \theta_\Gamma)$. Hence the statement (i) follows.

Write $\widehat{\Gamma} := \Gamma_{cdg}(A^!, (A^!)^*(-d-1))$. Then $\widehat{\Gamma}_d = A_d^! \oplus (A_1^!)^*$ and $\widehat{\Gamma}_{d+1} = \mathbb{k}$. Now it is clear that $\delta_{\Gamma_{cdg}}(\widehat{\Gamma}_d) = 0$. By Theorem 3.3, $A[z; \xi]$ is Calabi-Yau. Thus the statement (ii) follows from Lemma 5.3. \square

As a special case of Proposition 5.7, we obtain the following theorem.

Theorem 5.8. *Let A be a Koszul Calabi-Yau algebra of global dimension d , and let U be an arbitrary PBW-deformation of A . Assume that ζ is the Nakayama automorphism of U as in Proposition 5.5. Then $U[z; \zeta]$ is Calabi-Yau.*

Proof. Since A is Calabi-Yau, then the quadratic dual $A^!$ is graded symmetric; that is, the Nakayama automorphism of $A^!$ is $\varphi = \epsilon^{d+1}$. Then $\epsilon^{d+1}\varphi = id$, which is certainly an automorphism of the dual cdga $(A^!, \delta_{A^!}, \theta_{A^!})$ of the PBW-deformation U . \square

If U is an augmented PBW-deformation, then the ground field \mathbb{k} is a left U -module through the augmentation map. Let $E(U) := \bigoplus_{i \geq 0} \text{Ext}_U^i(U\mathbb{k}, U\mathbb{k})$. Note that the curvature element of the cdga $(A^!, \delta_{A^!}, \theta_{A^!})$ dual to U is zero. Thus $(A^!, \delta_{A^!})$ is a usual dga; that is, $\delta_{A^!}^2 = 0$. So, the cohomology $HA^!$ of $(A^!, \delta_{A^!})$ is a graded algebra.

Proposition 5.9. *Let A be a Koszul Calabi-Yau algebra, and U an augmented PBW-deformation of A . Assume that ζ is the Nakayama automorphism of U as in Proposition 5.5. Then*

$$E(U[z; \zeta]) \cong \Gamma(H(A^!), \epsilon^{d+1}H(A^!)^*(-d-1)).$$

Proof. If U is an augmented PBW-deformation of A , then $U[z; \zeta]$ is an augmented PBW-deformation of $A[z]$. Hence $\Gamma_{cdg}(A^!, (A^!)^*(-d-1))$ is a dga. By [PP, Proposition 6.1] and Proposition 5.7, $E(U[z; \zeta])$ is isomorphic to the cohomology algebra of the dga $\Gamma_{cdg}(A^!, (A^!)^*(-d-1))$. Now Lemma 1.3 implies the desired isomorphism. \square

In Proposition 5.7, we need the condition that the composition $\epsilon^{d+1}\varphi$ is an automorphism of the cdga $(A^!, \delta_{A^!}, \theta_{A^!})$. Certainly, there is no reason to expect that $\epsilon^{d+1}\varphi$ is always compatible with the

cdga structure on $A^!$. For example, if A is an AS-regular algebra of global dimension 2, then $A^!$ is of length 2. Hence any linear map $\delta : A_1^! \rightarrow A_2^!$ and any element $\theta \in A_2^!$ form a cdga $(A^!, \delta, \theta)$. Below, we show that the condition that $\epsilon^{d+1}\varphi$ is compatible with the cdga structure on $A^!$ is necessary in case that A is an AS-regular algebra of global dimension 2.

From now on, we assume that A is an AS-regular algebra of global dimension 2. Then $A \cong T(V)/(f)$ where V is an n -dimensional vector space with a fixed basis $\{x_1, \dots, x_n\}$, and $f = (x_1, \dots, x_n)M(x_1, \dots, x_n)^t$ with M an invertible $n \times n$ matrix [Z, DV]. Some properties of A has been listed below Corollary 3.4. In the following discussions, we keep the same notions as those in the items listed below Corollary 3.4. We chose a new basis $\{\omega_1, \dots, \omega_n\}$ of $A_1^!$ as:

$$(\omega_1, \dots, \omega_n) = (x_1^*, \dots, x_n^*)M^{-1}.$$

Then we have $x_i^*\omega_j = \delta_j^i\varpi$, where ϖ is the basis of $A_2^!$ as in the item (iii) below Corollary 3.4.

Let U be a PBW-deformation of A . Since A is a domain, as we pointed out earlier, there is a unique Nakayama automorphism. Hence we may say “the” Nakayama automorphism of U .

Theorem 5.10. *Let U be a PBW-deformation of A with ζ the Nakayama automorphism of U , and let $(A^!, \delta_{A^!}, \theta_{A^!})$ be the dual cdga of U . Assume $\delta_{A^!}(\omega_1, \dots, \omega_n) = (\lambda_1\varpi, \dots, \lambda_n\varpi)$. The following are equivalent:*

- (i) $\epsilon\varphi$ is an automorphism of the cdga $(A^!, \delta_{A^!}, \theta_{A^!})$;
- (ii) $U[z; \zeta]$ is Calabi-Yau;
- (iii) $(\lambda_1, \dots, \lambda_n)M = -(\lambda_1, \dots, \lambda_n)M^t$.

Proof. That (i) implies (ii) follows Proposition 5.7. (ii) \implies (iii). Let $B = A[z; \zeta]$, $\widehat{U} = U[z; \zeta]$, and $(B^!, \delta_{B^!}, \theta_{B^!})$ the dual cdga of \widehat{U} . Let $\Gamma := \Gamma(A^!, (A^!)^*(-3))$, and $(\Gamma, \delta_\Gamma, \theta_\Gamma)$ the cdga induced by $(B^!, \delta_{B^!}, \theta_{B^!})$ through the isomorphism Ψ given in (13) and (14). The equation (17) in the present case reads as follows:

$$(21) \quad \delta_\Gamma(0, \varpi^*) = (0, X),$$

where $X = -(\omega_1^*, \dots, \omega_n^*)(M^{-1})^t M(\lambda_1, \dots, \lambda_n)^t$. Now, since \widehat{U} is Calabi-Yau and B is obviously a domain, we have $\delta_\Gamma(\Gamma_2) = 0$ by Lemma 5.3. Hence we have

$$\begin{aligned} 0 &= \delta_\Gamma((\omega_i, 0) * (0, \varpi^*)) \\ &= \delta_\Gamma(\omega_i, 0) * (0, \varpi^*) - (\omega_i, 0) * \delta_\Gamma(0, \varpi^*) \\ &= (\lambda_i\varpi, 0) * (0, \varpi^*) - (\omega_i, 0) * (0, X) \\ &= (0, \lambda_i\varpi \cdot \varpi^*) - (0, \omega_i \cdot X) \\ &= (0, \lambda_i) - (0, \omega_i \cdot X), \end{aligned}$$

where the notion “ \cdot ” in $\varpi \cdot \varpi^*$ and $\omega_i \cdot X$ is the left $A^!$ -module action on $(A^!)^*$, and in the last identity, we identify \mathbb{k} with $(A_0^!)^*$. Thus we obtain

$$(22) \quad \lambda_i = \omega_i \cdot X$$

for all $i = 1, \dots, n$. Note that $\omega_i \cdot X = -(0, \dots, 0, 1, 0, \dots, 0)(M^{-1})^t M(\lambda_1, \dots, \lambda_n)^t$. From Equation (22), we obtain $(\lambda_1, \dots, \lambda_n)^t = -(M^{-1})^t M(\lambda_1, \dots, \lambda_n)^t$, and hence (iii) follows.

(iii) \implies (i). We have

$$\begin{aligned} (\epsilon\varphi)\delta_{A^!}(\omega_1^*, \dots, \omega_n^*) &= \epsilon\varphi(\lambda_1\varpi^*, \dots, \lambda_n\varpi^*) \\ &= (\lambda_1\varpi^*, \dots, \lambda_n\varpi^*); \end{aligned}$$

and

$$\begin{aligned} \delta_{A^!}(\epsilon\varphi)(\omega_1^*, \dots, \omega_n^*) &= -\delta_{A^!}\varphi(x_1^*, \dots, x_n^*)M^{-1} \\ &= -\delta_{A^!}(x_1^*, \dots, x_n^*)M^{-1}M^tM^{-1} \\ &= -(\lambda_1\varpi, \dots, \lambda_n\varpi)M^tM^{-1}. \end{aligned}$$

Now the condition (iii) insures that $(\epsilon\varphi)\delta_{A^!} = \delta_{A^!}(\epsilon\varphi)$. Therefore $\epsilon\varphi$ is an automorphism of cdga $(A^!, \delta_{A^!}, \theta_{A^!})$ since $A^!$ is of length 2. \square

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J.-W. HE

DEPARTMENT OF MATHEMATICS, SHAOXING COLLEGE OF ARTS AND SCIENCES, SHAOXING ZHEJIANG 312000, CHINA

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF ANTWERP, MIDDELHEIMLAAN 1, B-2020
ANTWERP, BELGIUM

E-mail address: `jwhe@usx.edu.cn`

F. VAN OYSTAEYEN

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF ANTWERP, MIDDELHEIMLAAN 1, B-2020
ANTWERP, BELGIUM

E-mail address: `fred.vanoystaeyen@ua.ac.be`

Y. ZHANG

DEPARTMENT WNI, UNIVERSITY OF HASSELT, UNIVERSITAIRE CAMPUS, 3590 DIEPENBEEK, BELGIUM

E-mail address: `yinhuo.zhang@uhasselt.be`